Hyperbolic PDEs and Finite-Volume Methods I

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Goal: Methods for solution of hyperbolic/mixed PDEs

Hyperbolic (or mixed) PDEs appear everywhere in physics and engineering: Euler equations, Navier-Stokes equations, MHD equations, Einstein's equation of general relativity, shallow-water equations, ...

- Describe phenomena that travel with *finite-speed*: clearly all fundamental physics must obey causality, hence hyperbolic systems are fundamental (though approximations of hyperbolic equations may violate finite-speed constraint).
- Display very rich structure shocks and other discontinuities, instabilities, turbulence ... and in most problems, a mix of all!
- Specialized methods are needed to solve such systems: naive algorithms can **cause disaster**!

My goal is to develop *conceptual* understanding about such equations and numerical methods to solve them. Literature is too vast to cover in few lectures!

No Free Lunch Principle

No Free Lunch Principle

There is no *unique* discrete system of equations corresponding to a given system of continuous equations. No discrete system is perfect and a method that works well in one situation may not work well in others.

"All numerical methods suck, though some suck less than others. Make sure your method sucks less that the competition"

Dissipation, dispersion and robustness

Typically, numerical methods trade between accuracy and robustness. Very accurate schemes are not too robust and highly robust schemes are typically not very accurate. Must find a balance.

- To capture shocks and sharp gradient features some localized diffusion (dissipation) is needed via *limiters* or some other means. Some dissipation may also be needed for stability.
- Dissipation typical leads to a loss of other properties: in particular, for Maxwell equations it can cause EM energy to *decay*. For Euler equations high-k modes may get over-damped due to limiters: not good for turbulence problems.
- An ideal situation is to apply dissipation only where it is needed and use low-dissipation schemes elsewhere. However, this is easier said than done. How to determine where to apply dissipation is very tricky, specially in nonlinear complex flows. Ease to confuse physical features for numerical artifacts (and vice-versa!)

Hyperbolic PDEs describe phenomena that travel at finite speed

An intuitive “definition” that we will initially work with before stating the mathematically rigorous definition:

Definition (Hyperbolic PDEs “Intuitive Definition”)

A hyperbolic PDE is one in which all phenomena travel at a finite speed.

Some prototypical examples we will look at more closely:

- Advection equation: simplest trivial *linear* hyperbolic equation. Trivial but very important!
- Maxwell equation of electromagnetism: linear hyperbolic system
- Euler equations: probably historically the most important *nonlinear hyperbolic system*. Basis of vast literature on numerical methods and basis for more complex equations: ideal MHD, (general) relativistic hydro/MHD, Navier-Stokes solvers, etc. Need to understand even if you want to follow literature and apply methods to your own problem.

Hyperbolic PDEs: rigorous definition, no reliance on linearization

Consider a system of conservation laws written as

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0.$$

where $\mathbf{Q}$ is a vector of conserved quantities and $\mathbf{F}(\mathbf{Q})$ is a vector of fluxes. This system is called *hyperbolic* if the flux Jacobian

$$\mathbf{A} \equiv \frac{\partial \mathbf{F}}{\partial \mathbf{Q}}$$

has *real eigenvalues* and a *complete set of linearly independent* eigenvectors. In multiple dimensions if $\mathbf{F}_i$ are fluxes in direction i then we need to show that arbitrary linear combinations $\sum_i n_i \partial \mathbf{F}_i / \partial \mathbf{Q}$ have real eigenvalues and linearly independent set of eigenvectors.

To compute eigensystem often easier to work in *quasilinear form*

To derive eigensystem it is sometimes easier to work in non-conservative (quasi-linear) form of equations. Start with

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0.$$

and introduce an invertible transform $\mathbf{Q} = \varphi(\mathbf{V})$ where $\mathbf{V}$ are some other variables (for example: density, velocity and pressure). Then the system converts to

$$\frac{\partial \mathbf{V}}{\partial t} + \underbrace{(\varphi')^{-1}\mathbf{A}\varphi'}_{\mathbf{B}} \frac{\partial \mathbf{V}}{\partial x} = 0.$$

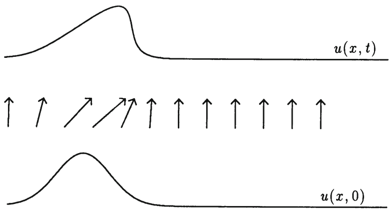
Can easily show eigenvalues of $\mathbf{A}$ are same as that of $\mathbf{B}$ and right eigenvectors can be computed from $\varphi'\mathbf{r}_p$ and left eigenvectors from $\mathbf{l}_p(\varphi')^{-1}$.

Burgers' equation

Simplest nonlinear scalar equation, has a quadratic nonlinearity:

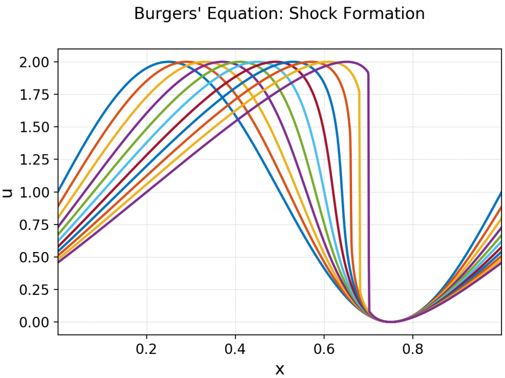
$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0.$$

Eigenvalue $\lambda = u$. Note that locally, eigenvalue depends on the solution itself: leads to a situation in which after some finite time to the formation of a *shock* as the "characteristics" will intersect.



Burgers' equation: shock formation

Due to varying characteristics speed (eigenvalue) the solution can "pile up" leading to the formation of a shock.



Weak-solutions and entropy conditions

At a shock the solution has a discontinuity. Hence, derivatives are not defined! Differential form of the equations break-down. We must use concept of weak-solutions in this case.

Let $\phi(x,t)$ is a compactly supported (i.e. zero outside some bounded region) smooth function (enough continuous derivatives). Then multiply conservation law

$$\int_0^\infty \int_{-\infty}^\infty \phi(x,t) \left[\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} \right] dx\, dt = 0$$

by $\phi(x,t)$ and integrating by parts to get the *weak-form*

$$\int_0^\infty \int_{-\infty}^\infty \left[\frac{\partial \phi}{\partial t} \mathbf{Q} + \frac{\partial \phi}{\partial x} \mathbf{F} \right] dx\, dt = -\int_{-\infty}^\infty \phi(x,0) \mathbf{Q}(x,0) dx\, dt.$$

Definition (Weak-solution)

A function $\mathbf{Q}(x,t)$ is said to be a weak-solution if it satisfies the weak-form for all compact, smooth $\phi(x,t)$.

Weak-solutions and entropy conditions

Unfortunately, weak-solutions are not unique! Why does this happen?

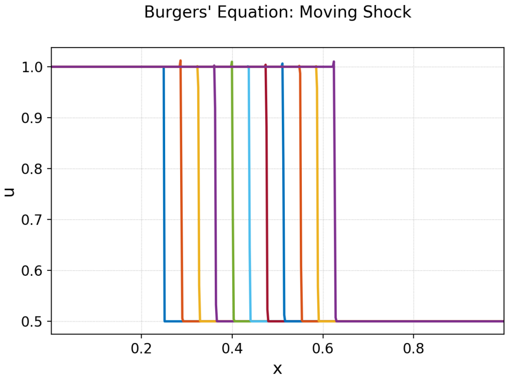
In physical problems there is always some non-ideal effects (viscosity, Landau damping etc) that does not allow a genuine discontinuity to form. However, this “viscous shock layer” can be extremely thin compared to system size. Also, we know entropy must increase in the physical universe.

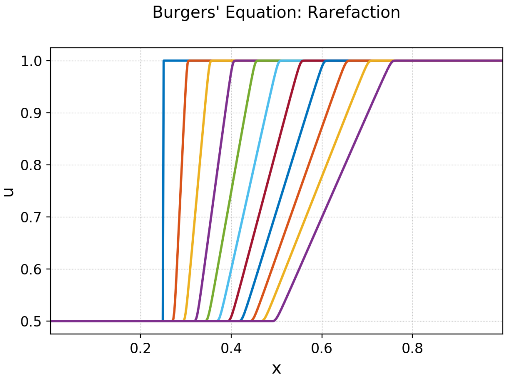
This indicates we can recover uniqueness in two ways

- Add a viscous (diffusion) term and take limit of viscosity going to zero. (Generally not convenient for numerical work)
- Impose *entropy condition*: construct an *entropy* function such that it remains conserved for smooth solutions but *increases* across a shock. Entropy is naturally suggested in most physical problems.

Weak-solutions and entropy conditions

When characteristics *diverge* (right plot below) the weak-solution is not unique. A false "shock" solution also is a weak-solution. Imposing *entropy condition* gives a *rarefaction* wave seen in the right plot.





Euler equations of invicid fluids

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \mathcal{E}\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\\ (\mathcal{E}+p)u\end{bmatrix}=0$$

Here $\mathcal{E}=p/(\gamma-1)+\rho u^2/2$ is the total energy. Eigenvalues of this system are $\{u-c_s, u, u, u, u+c_s\}$ where $c_s=\sqrt{\gamma p/rho}$ is the sound speed. See class notes for left/right eigenvectors.

Note: in the limit $p\to 0$ all eigenvalues become u and for cold-fluid ($p=0$) the system does not possess complete set of eigenvectors. (Cold fluid model is important to model dust, for example, in astrophysical systems or in say volcanic explosions).

Euler equations: transport of kinetic energy

The energy conservation equation for Euler equation is

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot [(\mathcal{E} + p)\mathbf{u}] = 0$$

where

$$\mathcal{E} = \underbrace{\frac{p}{\gamma - 1}}_{\mathrm{IE}} + \underbrace{\frac{1}{2}\rho u^2}_{\mathrm{KE}}.$$

We can derive instead a *balance law* (not conservation law) for transport of KE

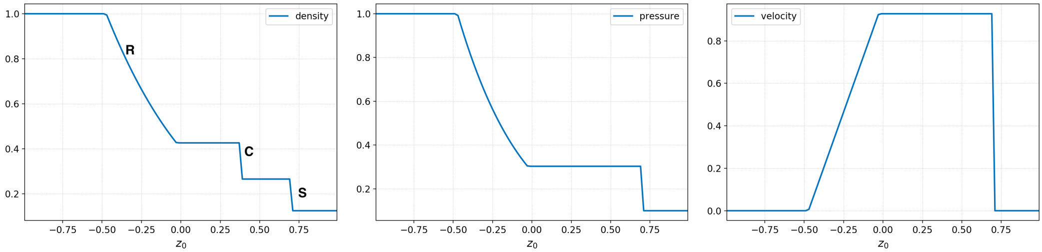
$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) + \nabla \cdot \left(\frac{1}{2}\rho u^2 \mathbf{u}\right) = -\mathbf{u} \cdot \nabla p + \frac{q}{m}\rho \mathbf{u} \cdot \mathbf{E}$$

For turbulence calculations it is important to ensure that in the *numerics* exchange of kinetic and internal energy (and in case of plasma field-particle energy) is only via the RHS terms (pressure work and work done by electric field).

Many shock-capturing and higher-order methods can mess this up for high-k (short wavelength) modes due leading to incorrect energy spectra. (No Free Lunch Principle).

Euler equations: shocks, rarefactions and contacts

In addition to shocks and rarefactions which we saw in Burgers's equation, Euler equations also support *contact discontinuities*, across which density has a jump but not pressure or velocity.



Ideal MHD equations

Ideal MHD equations are very important to both fusion and astrophysical problems. Written in non-conservative form they are

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} &= 0 \\
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} &= \frac{1}{\mu_0 \rho}(\nabla \times \mathbf{B}) \times \mathbf{B} \\
\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} &= 0 \\
\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) &= 0
\end{aligned}
$$

with the constraint $\nabla \cdot \mathbf{B} = 0$. The eigensystem is complicated to compute! (Try doing it yourself). Eigenvalues are $u \pm c_f$, $u \pm c_s$, $u \pm c_a$, and u (7 eigenvalues for 8 equations). Here c_f, c_s are the fast/slow magnetosonic speeds and c_a is the Alfven speed. See Ryu and Jones ApJ **442** 228-258, 1995 (linked on website).