Hyperbolic PDEs and Finite-Volume Methods. Homework Set I

Ammar Hakim

Summary

Intuitively, hyperbolic PDEs describe phenomena that propagate with finite speed. One way to check this is to *linearize* the equations around a uniform background and compute the dispersion relation for propagation of small amplitude waves. A system linearized p hyperbolic equations will have dispersion relations of the form $\omega^p = c_0^p k$ where c_0^p are the speeds computed from the equilibrium quantities. Note that the phase and group velocities for linear waves are the same for hyperbolic systems.

Definition 1 (Hyperbolic PDEs)**.** *Consider a system of conservation laws written as*

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0. \tag{1}$$

where $\mathbf{Q}$ is a vector of conserved quantities and $\mathbf{F}(\mathbf{Q})$ is a vector of fluxes. This system is called hyperbolic *if the flux Jacobian*

$$\mathbf{A} \equiv \frac{\partial \mathbf{F}}{\partial \mathbf{Q}}$$

has real eigenvalues *and a* complete set of linearly independent *eigenvectors. A hyperbolic system is called* strictly hyperbolic *if all eigenvalues are distinct.*

To derive eigensystem it is sometimes easier to work in non-conservative (quasi-linear) form of equations. Start with

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0.$$

and introduce an invertible transform $\mathbf{Q} = \varphi(\mathbf{V})$ where $\mathbf{V}$ are some other variables (for example: density, velocity and pressure). Then the system converts to

$$\frac{\partial \mathbf{V}}{\partial t} + \underbrace{(\varphi')^{-1}\mathbf{A}\varphi'}_{\mathbf{B}} \frac{\partial \mathbf{V}}{\partial x} = 0.$$

Can easily show eigenvalues of $\mathbf{A}$ are same as that of $\mathbf{B}$ and right eigenvectors can be computed from $\varphi' \mathbf{r}_p$ and left eigenvectors from $\mathbf{l}_p(\varphi')^{-1}$.

As the solution to hyperbolic PDEs can develop discontinuities we need to instead consider a broader class of solutions than those supported by the PDE (as derivatives are not defined at discontinuities). Instead we must use the concept of *weak-solutions*, defined below.

Definition 2 (Weak-solution)**.** *Let $\phi(x,t)$ be a compactly supported (i.e. zero outside some bounded region) smooth function (enough continuous derivatives). Then*

$$\int_0^\infty \int_{-\infty}^\infty \left[\frac{\partial \phi}{\partial t}\mathbf{Q} + \frac{\partial \phi}{\partial x}\mathbf{F}\right] dx\, dt = -\int_{-\infty}^\infty \phi(x,0)\mathbf{Q}(x,0) dx. \tag{2}$$

is called the weak-form *of the conservation law Eq. (1). A function $\mathbf{Q}(x,t)$ is said to be a weak-solution if it satisfies the weak-form for all compact, smooth $\phi(x,t)$.*

Cast of characters: three characters appear in the problems below: Ms. Fivo L. Hacker, Mr. Fido Node and Prof Symplectico. One can guess what point of view each of these characters represents. Also, for some of these problem you may/will need to use a computer algebra system.

Problem 1: Eigensystem of Euler Equations

Consider the Euler equations in 1D conservative form

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho u \\ E\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}\rho u \\ \rho u^2 + p \\ (E+p)u\end{bmatrix} = 0$$

where

$$E = \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1}.$$

Show that these equations can also be written in the *quasi-linear* form

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{V}}{\partial x} = 0$$

where $\mathbf{V} = [\rho, u, p]^T$ and $\mathbf{B}$ is a matrix you should determine. Compute the eigenvalues and right eigenvectors of $\mathbf{B}$ and use them to compute the eigenvectors of the flux Jacobian of the conservative system. Hence show that the Euler equations are indeed hyperbolic as long as $p > 0$ and $\rho > 0$.

Problem 2: The effect of oscillatory source terms

As discussed in class, source terms can significantly change the physics contained in the homogenous system. As an example consider the Euler equations with an oscillatory source as follows:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho u \\ \rho v \\ E\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}\rho u \\ \rho u^2 + p \\ \rho u v \\ (E+p)u\end{bmatrix} = \begin{bmatrix}0 \\ \rho v \Omega \\ -\rho u \Omega \\ 0\end{bmatrix}$$

where now

$$E = \frac{1}{2}\rho(u^2 + v^2) + \frac{p}{\gamma - 1}$$

and where Ω is a constant with units of inverse time. Linearize this system around a uniform non-flowing ($u_0 = v_0 = 0$) background ρ_0, p_0 and by considering solutions of the form $e^{-i\omega t}e^{ikx}$ derive the dispersion relation

$$\omega = \pm(k^2 c_{s0}^2 + \Omega^2)^{1/2}$$

where $c_{s0} = \sqrt{\gamma p_0/\rho_0}$. Clearly, this system only contains propagating but *dispersive* waves, however it is not hyperbolic as the dispersion relation is not *linear*. Derive the exact solution to the linearized system in terms of the perturbations of the x-component of the velocity $u_1(x,t)$.

Finally, consider the initial velocity field

$$u_1(x,0) = U_0 \sum_{n=0}^{N} \frac{i}{2n+1} e^{ik_n x}$$

for $x \in [0,1]$ and where $k_n = 2\pi(2n+1)$. For $N \to \infty$ this represents a step function. Take $N = 100$ and plot the exact solution of the perturbed density, $\rho_1(x,t)$, at $t = 1000$ for $\rho_0 = p_0 = 1$, $\gamma = 2$ and $\Omega = 10$. (If you make a movie of the exact solution in time you can see the initial spectrum of waves disperse and form complex patterns. For $\Omega = 0$ the initial condition will simply travel with sound speed as in this case the system becomes hyperbolic and the waves have the same group and phase velocities).

Problem 3: (Optional, challenging) Eigenvalues of ideal MHD equations

The ideal-MHD equations can be written in non-conservation law form

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} &= 0 \\
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{\nabla p}{\rho} &= \frac{1}{\mu_0 \rho}(\nabla\times\mathbf{B})\times\mathbf{B} \\
\frac{\partial p}{\partial t} + \mathbf{u}\cdot\nabla p + \gamma p\nabla\cdot\mathbf{u} &= 0 \\
\frac{\partial \mathbf{B}}{\partial t} - \nabla\times(\mathbf{u}\times\mathbf{B}) &= 0
\end{aligned}$$

with the constraint $\nabla\cdot\mathbf{B} = 0$. Write this equation in quasilinear form in 1D and compute the eigenvalues, showing they are all real if $\rho > 0$ and $p > 0$.

Problem 4: Shock solution for Burgers' equation

In class we wrote down the solution to Burgers' equation for initial condition $u(x,0) = u_l$ for $x < 0$ and $u(x,0) > u_r$ for $x > 0$ and where $u_l > u_r$. That solution was:

$$\begin{aligned}
u(x,t) &= u_l \quad x < st \\
u(x,t) &= u_r \quad x > st.
\end{aligned}$$

Show that this indeed is a weak-solution of the Burgers' equation, that is it satisfies Eq. (2). Recall that the Burgers' equation is a nonlinear scalar hyperbolic PDE:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0.$$

Problem 5: Reimann problem for Maxwell equations

Find the exact solution to the the Reimann problem for Maxwell equations in 1D:

$$\frac{\partial}{\partial t}\begin{bmatrix} E_y \\ E_z \\ B_y \\ B_z \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} c^2 B_z \\ -c^2 B_y \\ -E_z \\ E_y \end{bmatrix} = 0.$$

Note you do not actually need to compute the complete eigensystem of these equations, though that is one way to solve the problem.

Problem 6: L_2 norm and entropy

Prof Symplectico points out to Fivo L. Hacker that the advection equation for the distribution function $f(x, v, t)$ is a Hamiltonian system

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$$

with the Hamiltonian $H = v^2/2$. Hence, it has infinite number of Casimir invariants, including the L_2 norm

$$\frac{\partial}{\partial t}\left(\frac{1}{2}f^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}vf^2\right) = 0.$$

that must be preserved (even locally) by a good numerical scheme. Prove that this expression is indeed true but point out, as Fivo would, the key assumption used in deriving this expression and why it may not be satisfied. Instead, to achieve stability she has designed a scheme that *decays* the L_2 norm, i.e.

$$\frac{d}{dt}\int \frac{1}{2}f^2 \, dx \, dv \leq 0.$$

Using the identity $\ln(y) \leq y - 1$ show that Fivo's scheme, somewhat reassuringly, *increases* the entropy, i.e.

$$\frac{d}{dt}\int -f \ln(f) \, dx \, dv \geq 0.$$

Problem 7: Changing representations

In class we stated the principle "when studying or designing numerical schemes **never** confuse one solution representation for another." Unfortunatley, Fivo L. Hacker's friend Fido Node has given her initial conditions f_i, $i = 1, \ldots, N$ evaluated at cell centers. How will she convert this data to *sufficiently high order* cell-average values to use in her third-order finite-volume code?

Fivo also would like to compute the following integral

$$E = \sum_j \int_{I_j} f g \, dx$$

As she learned in class this is not so simple as, in general, $(fg)_i \neq f_i g_i$ (that is, the cell-average of the product is not the product of cell-averages). How can this integral be computed carefully such that the underlying order of her scheme (say third order) can be preserved? Derive the stencil to do so.